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# ON THE STABILITY OF ELASTIC COMPRESSIBLE BODIES UNDER ALL-AROUND COMPRESSION

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The stability of simply-connected isotropic compressible elastic bodies with an arbitrary form of the elastic potential subjected to uniform all-around compression is investigated. The three-dimensional linearized theory of elastic stability for finite and small precritical deformations is involved. The case is considered when the body surface consists of two parts, one of which is rigidly clamped or hinged. It is proved that the equilibrium state will be stable if the pressure is applied in the form of a "follower" load on the second part of the surface, and is unstable if the pressure is applied in the form of a "dead" load on this part of the surface. In the latter case, the critical load for thin-walled bodies is approximately half the Euler force. Examples are considered for rectangular and circular plates, as well as for a circular rod in the case of materials with different forms of the elastic potential.

After the paper [1] had been published, the question of the stability of isotropic compressible simply-connected bodies under all-around compression was considered in numerous papers from different aspects of the three-dimensional theory of stability under small and finite precritical deformations. The disagreement between the results from these theories was explained by the inaccuracies of the theory of small precritical deformations. Results have been obtained in [2, 3] in general form for the theory of finite and small precritical deformation, and a survey of investigations on the problem considered is presented in [2].

**1. Formulation of the problem.** Let us examine two kinds of problems of the theory of elasticity. Let problems with identical boundary conditions on the whole body surface be among the first kind. In this case, it is shown in [2] that under the effect of follower loads the equilibrium state will be stable if the conditions

$$\lambda_0 + {}^2/_3\,\mu_0 > 0, \quad \mu_0 > 0$$

are satisfied.

The quantities $\lambda_0$ and $\mu_0$ are expressed in terms of the elastic potential [2, 4]. Since the inequalities presented assure an explanation of phenomena observed experimentally, we shall consider that they are always satisfied and we shall consider them as constraints on the form of the elastic potential.

Such conditions have also been obtained in [5] and in a number of other papers. The peculiarity in the results in [2] is the fact that the stability conditions are obtained in general form for the theory of finite and small precritical deformations in the latter case for which it was necessary to use a more exact expression [2] than is ordinarily taken [6, 8] in the theory of small precritical deformations to determine the follower loads. In the case of the action of dead loads, it is shown in [3] with expamples of the plane problem of a circular continuous cylinder and the axisymmetric problem for a continuous sphere that there exist critical loads which do not contradict the inequalities presented.

Among the second kind are problems with different boundary conditions on separate parts of the boundary surface of the body. In this case, it is shown in [2] with the example of a hinge-supported strip that the equilibrium state will be stable if a follower load is applied to the side surfaces, and unstable if a dead load is applied to the side surfaces. In this latter case, the calculated critical load for a thin-walled strip turned out to be considerably less than the Euler force under axial compression.

Let us consider spatial problems on the stability of plates and cylindrical bodies under uniform all-around compression when different boundary conditions are given on separate parts of the boundary surface. We shall consider the body compressible, isotropic with an arbitrary form of the elastic potential, and simply-connected, which will assure the existence of a homogeneous precritical state. We shall use a Lagrange coordinate system which will coincide with the Cartesian $(x_1, x_2, x_3)$ or the circular cylindrical $(r, \theta, x_3)$ systems in the unstrained state. We shall indicate quantities referring to the precritical state by the superscript zero. We conduct the investigation in general form for three-dimensional linearized theories of elastic stability under finite and small precritical strains [8, 9].

According to [2, 3], the linearized equations of motion in the absence of volume force perturbations can be represented as follows for the case under consideration:

$$(\lambda_0 + 2\mu_0) \operatorname{grad} \operatorname{div} \mathbf{u} - \mu_0 \operatorname{rot} \operatorname{rot} \mathbf{u} - \rho \mathbf{u}^{\cdot\cdot} = 0 \tag{1.1}$$

The stress boundary conditions on the part $S_1$ of the body surface can be written as

$$\mathbf{Q}|_{S_1} = \mathbf{P}; \quad \mathbf{Q} \equiv \mathbf{N}(\lambda_0 + \sigma_0^*) \operatorname{div} \mathbf{u} + (2\mu_0 - \sigma_0^*)(\mathbf{N}\nabla)\mathbf{u} + (\mu_0 - \sigma_0^*)\mathbf{N} \times \operatorname{rot} \mathbf{u} \tag{1.2}$$

Here $\mathbf{N}$ is the normal direction to the body surface in the unstrained state, and $\mathbf{P}$ is the perturbation of the external loads acting on $S_1$.

The displacement boundary conditions on the part $S_2$ of the body surface can be written in the form

$$\mathbf{u}|_{S_2} = 0 \tag{1.3}$$

In the case of the action of dead loads $\mathbf{P} = 0$, in the case of the action of follower loads, the following expression has been obtained in [2]

$$\mathbf{P} = \sigma_0^* [\mathbf{N} \operatorname{div} \mathbf{u} - (\mathbf{N}\nabla) u - \mathbf{N} \times \operatorname{rot} \mathbf{u}]|_{S_1} \tag{1.4}$$

Expressions to determine the quantities $\lambda_0$, $\mu_0$ and $\sigma_0^*$ in terms of the elastic potential are presented in [2, 4] for different problem formulations.

The general solution of (1. 1) without inertial terms and taking account of the results in [8] for the case under consideration can be represented as

$$u_n = \frac{\partial}{\partial S}\psi - \frac{\partial^2}{\partial n \partial x_3}\chi, \quad u_S = -\frac{\partial}{\partial n}\psi - \frac{\partial^2}{\partial S \partial x_3}\chi \qquad (1.5)$$
$$u_3 = \frac{\lambda_0 + 2\mu_0}{\lambda_0 + \mu_0}\left(\Delta - \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}\frac{\partial^2}{\partial x_3^2}\right)\chi$$

where $\psi$ and $\chi$ are, respectively, the harmonic and biharmonic functions.

The expressions (1. 5) are written for a cylindrical body whose axis coincides with the axis $Ox_3$ and $n$ and $s$ are the normal and tangent to the cross-sectional outline.

It must be noted that (1. 4), which determines the follower load in the case of the theory of small precritical deformations differs from the corresponding expressions in [6 —8]. The expression (1. 4) has been found in [2] for the theory of small precritical deformations from the appropriate linearized relationships following from the theory of finite precritical deformations.

2. Certain general questions. Let us examine certain general questions referring to the class of three-dimensional stability problems under consideration.

On the applicability of the Euler method. Let us examine the question of the possibility of applying the Euler method to investigate the stability of elastic isotropic bodies under all-around compression when the boundary conditions are different on separate parts of the boundary surface (problems of the second kind). This question evidently does not occur in the case of the action of dead loads. Let us consider the case when the follower load is given on the part $S_1$ of the body surface bounded by the curve $L$. By using (1. 4) to determine the follower load, it is shown in [10] that the sufficient conditions for applicability of the Euler method are satisfied if one of the following quantities vanishes on the curve $L$: 1) the displacement directed along the normal to the surface $S_1$: 2) the displacement directed along the normal to the curve $L$ on the surface $S_1$.

Henceforth, we shall consider the case when the body surface consists of two parts $S = S_1 + S_2$, where the follower load acts on the surface $S_1$, i. e., the right sides of the boundary conditions (1. 2) have the form (1. 4). We shall also consider the surfaces $S_1$ and $S_2$ to intersect on the curve $L$.

Let us note that two cases are known when the Euler method can be used: the first is when a dead load acts, and the second is when a follower load acts on the whole body surface. Let us mention two other cases of the applicability of the Euler method under uniform all-around compression when the follower load act on the part $S_1$ of the body surface.

The first is when the part $S_2$ of the body surface is rigidly clamped, i. e., conditions (1. 3) are satisfied. These consitions are also satisfied on the curve $L$, therefore, the conditions obtained in [10] are also satisfied. Hence, the Euler method can ben applied independently of the body shape for the problem under consideration.

The second is when the surface $S_1$ and $S_2$ are orthogonal and the hinged-support conditions are satisfied in the integral sense, i. e.,

$$u_s = 0, \quad Q_n = 0 \tag{2.1}$$

Here $u_s$ is the component of the displacement vector in the tangent plane to $S_2$ and $Q_n$ is the component of the stress vector directed along the normal to $S_2$. Because of the orthogonality of $S_1$ and $S_2$, the first conditions (2.1) assures compliance with the first conditions obtained in [10].

Let us note that the sufficient conditions for the applicability of the Euler method [10] are also satisfied under conditions (2.1) when the normals to the surfaces $S_1$ and $S_2$ coincide on the curve $L$. In this case, the first condition in (2.1) assures compliance with the second condition in [10], however, this case has no clear physical interpretation.

It must be noted that a case when the follower load in the form of a uniform pressure is given on a part of the body surface is also investigated in [7]. However, a less accurate expression, compared to (1.4) is used to define the follower load in [7], hence, the Kirchhoff—Love hypothesis must be used. In this connection, the results in [7] refer just to the stability theories of thin-walled systems constructed by involving the Kirchhoff—Love hypothesis.

Stability for rigid clamping of the boundary surface. Let us examine the case when the part $S_1$ of the body surface $S = S_1 + S_2$ is loaded by a follower load while the part $S_2$ is rigidly clamped. The conditions (1.2) are satisfied on $S_1$ with (1.4) taken into account, while conditions (1.3) are satisfied on $S_2$. The Euler method is applicable in this case. Therefore, the problem under consideration reduces to (1.1) without inertial terms and to boundary conditions (1.2) and (1.3) taking (1.4) into account. The problem (1.1) —(1.4) goes over into the classical linear homogeneous problem of elasticity theory if $\lambda$ and $\mu$ are replaced by the parameters $\lambda_0$ and $\mu_0$ in the latter. In this case, as is known [5], the linear homogeneous problem of classical elasticity theory has a trivial solution if the above-mentioned stability conditions are satisfied; the equilibrium position will be stable independently of the body shape. When a dead load acts on $S_1$ it is impossible to obtain the stability condition in the form independent of the body shape since the boundary conditions (1.2) for zero right sides do not agree with the corresponding homogeneous boundary conditions of linear elasticity theory.

Stability for hinged support of part of the boundary surface. Let us consider a cylinder or arbitrary cross-section, whose axis is directed along $Ox_3$. Let us consider the side surface loaded by a follower load, while the endfaces $(x_3 = 0,\ x_3 = l)$ are hinge supported. In this case, according to (1.2) and (2.1), we obtain the following conditions for $x_3 = 0$ and $x_3 = l$:

$$u_1 = 0; \quad u_2 = 0; \quad (\lambda_0 + \sigma_0{}^*) \operatorname{div} \mathbf{u} + (2\mu_0 - \sigma_0{}^*)\partial u_3 / \partial x_3 = 0 \tag{2.2}$$

We obtain boundary conditions (1.2) on the side surface under the condition (1.4). The Euler method is applicable in the case under consideration. Therefore, the problem under consideration reduces to (1.1) without inertia terms, to boundary conditions (1.2) on the side surface with (1.4) taken into account and boundary conditions (2.2) on the endfaces for $x_3 = 0$ and $x_3 = l$. It is impossible to reduce the problem (1.1), (1.2), (1.4) and (2.2) to a linear homogeneous problem of classical elasticity theory by replacing the parameters $\lambda$ and $\mu$ by $\lambda_0$ and $\mu_0$ because of the structure of the last condition in (2.2). In this connection, we represent the displacements in the form

$$u_1 = w_1(x_1, x_2)\sin(\pi m l^{-1} x_3), \quad u_2 = w_2(x_1, x_2)\sin(\pi m l^{-1} x_3) \qquad (2.3)$$
$$u_3 = w_3(x_1, x_2)\cos(\pi m l^{-1} x_3)$$

The displacements (2.3) satisfy the hinge support conditions in the form (2.2),(2.3). From (1.1), (1.2), (1.4)and (2.3) we obtain a two-dimensional homogeneous problem in $w_i(x_1, x_2)$ which agrees with the appropriate homogeneous linear problem of classical elasticity theory upon replacement of the parameters $\lambda$ and $\mu$ by $\lambda_0$ and $\mu_0$. As is known, this latter problem has a unique trivial solution. Therefore, the equilibrium state under consideration is stable for a cylinder of arbitrary cross-section.

In the case of dead loads acting on $S_1$, the investigation must be performed with the specific shape of the body taken into account.

3. Plate stability. Let us examine the stability of rectangular and circular plates under hinged support, as well as in the rigid clamping case.

Rectangular plates. Let us consider a rectangular plate $(0 \leqslant x_1 \leqslant a;\ 0 \leqslant x_2 \leqslant b;\ -h \leqslant x_3 \leqslant +h)$ under all-around compression when it is hinge supported at $x_1 = 0$, $x_1 = a$, $x_2 = 0$ and $x_2 = b$ and is loaded by a dead load for $x_3 = \pm h$. In conformity with (2.1), (1.2) and (2.2), we obtain the boundary conditions in the form

$$x_1 = 0;\quad x_1 = a,\quad u_2 = 0;\quad u_3 = 0;\quad (\lambda_0 + \sigma_0^*)\,\mathrm{div}\,\mathbf{u} + (2\mu_0 - \sigma_0^*)\partial u_1 / \partial x_1 = 0 \qquad (3.1)$$

$$x_2 = 0;\quad x_2 = b,\quad u_1 = 0;\quad u_3 = 0;\ (\lambda_0 + \sigma_0^*)\,\mathrm{div}\,\mathbf{u} + (2\mu_0 - \sigma_0^*)\partial u_2 / \partial x_2 = 0 \qquad (3.2)$$

$$x_3 = \pm h,\ (2\mu_0 - \sigma_0^*)\frac{\partial u_1}{\partial x_3} + (\mu_0 - \sigma_0^*)\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) = 0 \qquad (3.3)$$
$$(2\mu_0 - \sigma_0^*)\frac{\partial u_2}{\partial x_3} + (\mu_0 - \sigma_0^*)\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = 0$$
$$(\lambda_0 + \sigma_0^*)\,\mathrm{div}\,\mathbf{u} + (2\mu_0 - \sigma_0^*)\,\partial u_3 / \partial x_3 = 0$$

The Euler method is applicable in the case under consideration. The general solution (1.5) has the form

$$u_1 = \frac{\partial}{\partial x_1}\psi - \frac{\partial^2}{\partial x_1 \partial x_3}\chi;\quad u_2 = -\frac{\partial}{\partial x_1}\psi - \frac{\partial^2}{\partial x_2 \partial x_3}\chi \qquad (3.4)$$
$$u_3 = \frac{\lambda_0 + 2\mu_0}{\lambda_0 + \mu_0}\left(\Delta - \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}\frac{\partial^2}{\partial x_3^2}\right)\chi$$

We represent the harmonic and biharmonic functions $\psi$ and $\chi$ satisfying conditions (3.1) and (3.2) for the bending buckling mode in the form

$$\psi = 4\ \mathrm{sh}\ \gamma x_3 \cos(\pi m a^{-1} x_1)\cos(\pi n b^{-1} x_2) \qquad (3.5)$$
$$\gamma^2 = (\pi m/a)^2 + (\pi n/b)^2$$
$$\chi = (B\ \mathrm{ch}\ \gamma x_3 + C\gamma x_3\ \mathrm{sh}\ \gamma x_3)\sin(\pi m a^{-1} x_1)\sin(\pi n b^{-1} x_2)$$

For buckling with the formation of a neck, the formations $\psi$ and $\chi$ satisfying conditions (3.1) and (3.2) are represented as follows

$$\psi = A \operatorname{ch} \gamma x_3 \cos (\pi m a^{-1} x_1) \cos (\pi n b^{-1} x_2) \tag{3.6}$$
$$\chi = (B \operatorname{sh} \gamma x_3 + C\gamma x_3 \operatorname{ch} \gamma x_3) \sin (\pi m a^{-1} x_1) \sin (\pi n b^{-1} x_2)$$

From (3.3) and (3.5), we obtain the characteristic equation in the form

$$\det \| a_{ij} \| = 0 \quad (i, j = 1, 2, 3) \tag{3.7}$$
$$a_{11} = -\mu_0 \pi n b^{-1} \gamma \operatorname{ch} \gamma h, \quad a_{12} = -\mu_0 \pi m a^{-1} \gamma^2 (2\mu_0 - \sigma_0^*) \operatorname{ch} \gamma h$$
$$a_{13} = -\pi m a^{-1} \gamma^2 \left[ (2\mu_0 - \sigma_0^*) \gamma h \operatorname{sh} \gamma h + 2\mu_0 \frac{\lambda_0 + \sigma_0^*}{\lambda_0 + \mu_0} \operatorname{ch} \gamma h \right]$$
$$a_{21} = -a_{11} m n^{-1} a^{-1} b; \quad a_{22} = a_{12} n m^{-1} b^{-1} a; \quad a_{23} = a_{13} n m^{-1} b^{-1} a$$
$$a_{31} = 0, \quad a_{32} = -(2\mu_0 - \sigma_0^*) \gamma^3 \operatorname{sh} \gamma h; \quad a_{33} = -(2\mu_0 - \sigma_0^*) \times$$
$$\gamma^3 \gamma h \operatorname{ch} \gamma h + \frac{2\mu_0^2 + \sigma_0^* (\lambda_0 + \mu_0)}{\lambda_0 + \mu_0} \gamma^3 \operatorname{sh} \gamma h$$

and from (3.7) we find

$$\det \| a_{ij} \| = -\gamma^8 \mu_0 (2\mu_0 - \sigma_0^*)^2 \gamma h \times \left\{ 1 - \frac{2(\lambda_0 + \mu_0)\mu_0 + \sigma_0^* (\lambda_0 + 3\mu_0)}{(\lambda_0 + \mu_0)(2\mu_0 - \sigma_0^*)} \frac{\operatorname{sh} 2\gamma h}{2\gamma h} \right\} \tag{3.8}$$

It follows from (1.2) and (1.4) that the boundary conditions in the case of a follower load are obtained formally from the boundary conditions for a dead load (conditions (1.2) for $\mathbf{P} = 0$) if we set $\sigma_0^* = 0$ in the latter. In this case we obtain from (3.8):

$$\det \| a_{ij} \| = -4\gamma^8 \mu_0^3 \gamma h \, [1 - \operatorname{sh} 2\gamma h / 2\gamma h] \tag{3.9}$$

Since sinh $2\gamma h > 2\gamma h$, we obtain $\det \| a_{ij} \| < 0$ from (3.9). In the case of the action of a follower load, the equilibrium state is therefore stable. The deduction obtained illustrates the general result of Sect. 2, obtained for a body of arbitrary shape.

For a dead load, we obtain a characteristic equation from (3.8), whose roots have physical meaning

$$1 - \frac{2\mu_0 (\lambda_0 + \mu_0) + \sigma_0^* (\lambda_0 + 3\mu_0)}{(\lambda_0 + \mu_0)(2\mu_0 - \sigma_0^*)} \frac{\operatorname{sh} 2\gamma h}{2\gamma h} = 0 \tag{3.10}$$

It must be noted that (3.10) agrees in form with the corresponding equation [2] for a strip, however, in the case under consideration the quantities $\lambda_0$, $\mu_0$, and $\sigma_0^*$ are expressed in terms of the elastic potential by using relationships for the spatial problem. Let us note that we obtain an equation in the form (3.10) also in the case of buckling with the formation of a neck.

Circular plates. Let us consider a circular plate $(0 \leqslant r \leqslant R;\ -h \leqslant x_3 \leqslant +h)$ under all-around compression when it is rigidly clamped for $r = R$ and subjected to a dead load for $x_3 = \pm h$. We form the rigid clamping condition for $r = R$ for the axisymmetric problem in the integral sense as follows:

$$u_r = 0, \quad \partial u_3 / \partial r = 0 \tag{3.11}$$

Henceforth, we shall limit ourselves to an investigation of just the axisymmetric problem of the plate. In this case, according to (1.2), we obtain the following boundary conditions for $x_3 = \pm h$

$$(2\mu_0 - \sigma_0^*)\partial u_r / \partial x_3 + (\mu_0 - \sigma_0^*)(\partial u_3 / \partial r - \partial u_r / \partial x_3) = 0 \tag{3.12}$$
$$(\lambda_0 + \sigma_0^*)(\partial u_r / \partial r + u_r / r + \partial u_3 / \partial x_3) + (2\mu_0 - \sigma_0^*)\partial u_3 / \partial x_3 = 0$$

The Euler method is also applicable in the case of the boundary conditions under consideration. Let us use the representation (1.5) of the solution of (1.1) without inertial terms. For the axisymmetric problem we obtain the following representation from (1.5)

$$u_r = \frac{\partial^2}{\partial r \partial x_3}\chi, \quad u_\theta = 0$$
$$u_3 = \frac{\lambda_0 + 2\mu_0}{\lambda_0 + \mu_0}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\mu_0}{\lambda_0 + 2\mu_0}\frac{\partial^2}{\partial x_3^2}\right)\chi \tag{3.13}$$

Here $\chi$ is a biharmonic function. We select the biharmonic function satisfying conditions (3.1) in the case of the bending buckling mode as

$$\chi = (A \operatorname{ch} \gamma x_3 + B\gamma x_3 \operatorname{sh} \gamma x_3)J_0(\gamma r), \quad \gamma = \varkappa_k / R, \quad J_0'(\varkappa_k) = 0 \tag{3.14}$$

and in the case of buckling with neck formation as

$$\chi = (A \operatorname{sh} \gamma x_3 + B\gamma x_3 \operatorname{ch} \gamma x_3)J_0(\gamma r) \tag{3.15}$$

Here $J_0(\gamma r)$ is the zero order Bessel function of the first kind. Let us first consider the bending buckling mode. From (3.14) and (3.12) we obtain the characteristic equation in the form (3.7) for $i, j = 1, 2$. The elements of the characteristic determinant are

$$\alpha_{11} = -\gamma^3(2\mu_0 - \sigma_0^*) \operatorname{ch} \gamma h, \quad \alpha_{12} = -\gamma^3[(2\mu_0 - \sigma_0^*)\gamma h \operatorname{sh} \gamma h + 2\mu_0(\lambda_0 + \sigma_0^*)(\lambda_0 + \mu_0)^{-1} \operatorname{ch} \gamma h], \quad \alpha_{21} = -\gamma^3 (2\mu_0 - \sigma_0^*) \operatorname{sh} \gamma h \tag{3.16}$$
$$\alpha_{22} = -\gamma^3[(2\mu_0 - \sigma_0^*)\gamma h \operatorname{ch} \gamma h - [2\mu_0^2 + \sigma_0^*(\lambda_0 + \mu_0)] \times (\lambda_0 + \mu_0)^{-1} \operatorname{sh} \gamma h]$$

We obtain the following expression from (3.7) and (3.16)

$$\det \| \alpha_{ij} \| = \gamma^6 (2\mu_0 - \sigma_0^*)^2 \gamma h \times \left[1 - \frac{2(\lambda_0 + \mu_0)\mu_0 + \sigma_0^*(\lambda_0 + 3\mu_0)}{(\lambda_0 + \mu_0)(2\mu_0 - \sigma_0^*)}\frac{\operatorname{sh} 2\gamma h}{2\gamma h}\right] \tag{3.17}$$

It must be noted that the solution (3.15) in the case of buckling with neck formation can be obtained from the solution (3.14) in the case of the bending buckling mode if the sinh and cosh are interchanged in the latter, which permits obtaining a characteristic equation for buckling with neck formation from (3.17). Since these functions enter symmetrically into (3.17), thus (3.17) refers to the two cases under consideration.

Setting $\sigma_0^* = 0$ in (3.17), we obtain the characteristic determinant for the follower load in the form

$$\det \| \alpha_{ij} \| = \gamma^6 4\mu_0^2 \gamma h \, (1 - \mathrm{sh}\, 2\gamma h \, / \, 2\gamma h)) \tag{3.18}$$

By analogy with the rectangular plate, we find that the equilibrium state is stable under the action of a follower load. The deduction obtained illustrates the general result in Sect. 2 obtained for a body of arbitrary shape.

For the case of a dead load, we obtain an equation in the form (3.10) whose roots have physical meaning from (3.17).

Now, let us consider examples for bodies with elastic potentials of specific form.

Example 1. Within the framework of the second variant of the theory of small precritical deformations, let us consider the example of a body with an elastic potential in the form

$$\Phi^\circ = {}^1/_2 \lambda A_1^{\circ 2} + \mu A_2^\circ \tag{3.19}$$

The potential (3.19) corresponds to a linearly elastic body, where $\lambda$ and $\mu$ are Lamé constants. From (3.19) and [2] we obtain

$$\lambda_0 = \lambda - \sigma_0, \quad \mu_0 = \mu + \sigma_0 \tag{3.20}$$

Analogously to [2], we determine two roots from (3.20) and (3.10)

$$\begin{aligned} p_{1,2} &= G \pm [G^2 - \mu(\lambda + \mu)(1 - 2\gamma h/\mathrm{sh}\, 2\gamma h)]^{1/2} \\ G &= {}^1/_4 \, [3\lambda + 5\mu - 2\gamma h \, (\lambda + \mu)/\mathrm{sh}\, 2\gamma h] \end{aligned} \tag{3.21}$$

Here $p = -\sigma_0$ is the compressive load intensity. From (3.21), we find for the long-wavelength buckling mode (for thin-walled plates)

$$p_c \approx \frac{1}{2} p_e \left[ 1 - (\gamma h)^2 \frac{14 - 23\nu + 14\nu^2}{30(1-\nu)^2} \right], \quad p_e = \frac{1}{3} (\gamma h)^2 \frac{E}{1-\nu^2}.$$

Here $p_c$ is the critical load, $p_e$ is the value of the critical load evaluated by involving the Kirchhoff—Love hypothesis for compression of a plate in its plane by a uniformly distributed load (Euler force); for a rectangular plate $\gamma_1^2 = \pi^2 (a^{-2} + b^{-2})$, for a circular plate $\gamma_1 = \varkappa_1 R^{-1}$.

Example 2. Within the framework of the theory of finite precritical deformations, let us consider the example for a body with a potential of harmonic type [11]:

$$\begin{aligned} &\Phi^\circ = {}^1/_2 \lambda S_1^{\circ 2} + \mu S_2^\circ, \quad S_1^\circ = (\lambda_1 - 1) + (\lambda_2 - 1) + (\lambda_3 - 1) \\ &S_2^\circ = (\lambda_1 - 1)^2 + (\lambda_2 - 1)^2 + (\lambda_3 - 1)^2 \end{aligned} \tag{3.22}$$

Here $\lambda_i$ are the elongation coefficients along the principal axes. From (3.22) and [4] we obtain

$$\lambda_0 = \lambda - (3\lambda + 2\mu)(\lambda_1 - 1) / \lambda_1, \quad \mu_0 = [2\mu + (3\lambda + 4\mu)(\lambda_1 - 1)] / (2\lambda_1) \qquad (3.23)$$
$$\sigma_0^* = (3\lambda + 2\mu)(\lambda_1 - 1) / \lambda_1$$

We find two roots from (3.23) and (3.10)

$$(\lambda_1)_1 = 0, \quad (\lambda_1)_2 = \left[(3\lambda + 2\mu)\, 2\,(\lambda + 2\mu)\frac{\operatorname{sh} 2\gamma n}{2\gamma h}\right] \times \left[\mu\lambda + (6\lambda^2 + 8\mu^2 + 15\lambda\mu)\frac{\operatorname{sh} 2\gamma h}{2\gamma h}\right]^{-1}$$

The first root has no physical meaning. In the case of the long-wavelength buckling mode, we obtain from the second expression

$$\lambda_1 = (\lambda_1)_c \approx 1 - {}^2/_3\,(\gamma_1 h)^2 \mu\,(\lambda + \mu)(\lambda + 2\mu)^{-1}\,(3\lambda + 2\mu)^{-1}$$

Therefore, buckling is possible under the effect of a dead load.

4. **Stability of a circular cylinder.** Let us consider the stability of a circular cylinder $(0 \leqslant r \leqslant R;\ 0 \leqslant x_3 \leqslant l)$ under all-around compression when it is hinge supported for $x_3 = 0$ and $x_3 = l$ but is loaded by a dead load for $r = R$. We understand the hinge-support conditions in the integral sense. For $x_3 = 0$ and $x_3 = l$ the boundary conditions have the form (2.2) and the form (1.2) for $r = R$ with $\mathbf{P} = 0$. The Euler method is applicable in the case under consideration. From (1.5), we obtain the general solution in the following form

$$u_r = \frac{1}{r}\frac{\partial}{\partial\theta}\psi - \frac{\partial^2}{\partial r \partial x_3}\chi, \quad u_\theta = -\frac{\partial}{\partial r}\psi - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial x_3}\chi \qquad (4.1)$$
$$u_3 = \frac{\lambda_0 + 2\mu_0}{\lambda_0 + \mu_0}\left(\Delta - \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}\frac{\partial^2}{\partial x_3^2}\right)\chi$$

Let us select the functions $\psi$ and $\chi$ satisfying the conditions (2.2) in the form

$$\psi = \sin(\pi m l^{-1} x_3)\, A I_n(m\pi l^{-1} r) \sin n\theta \qquad (4.2)$$
$$\chi = \cos \pi \frac{m}{l} x_3 \left[B I_n\left(m\frac{\pi}{l}r\right) + C m \frac{\pi}{l} r I_{n+1}\left(m\frac{\pi}{l}r\right)\right]\cos n\theta$$

We find the characteristic determinant from (4.2), (4.1) and (1.2)

$$\delta = (2\mu_0 - \sigma_0^*) R^{-7} m^3\alpha^3 \det \| \alpha_{ij} \| \quad (i,\ j = 1, 2, 3), \quad \alpha = \pi R / l \qquad (4.3)$$
$$\alpha_{11} = n\,(2\mu_0 - \sigma_0^*)[m\alpha I_n'(m\alpha) - I_n(m\alpha)]$$
$$\alpha_{12} = m^2\alpha^2 I_n''(m\alpha)$$
$$\alpha_{13} = -2\mu_0\frac{\lambda_0 + \sigma_0^*}{\lambda_0 + \mu_0} I_n(m\alpha) + (2\mu_0 - \sigma_0^*) \times [m\alpha I_{n+1}''(m\alpha) + 2I_{n+1}'(m\alpha)]$$
$$\alpha_{21} = -m^2\alpha^2(2\mu_0 - \sigma_0^*) I_n''(m\alpha) + (\mu_0 - \sigma_0^*) m^2\alpha^2 I_n(m\alpha)$$
$$\alpha_{22} = n[-m\alpha I_n'(m\alpha) + I_n(m\alpha)], \quad \alpha_{23} = -n\,(2\mu_0 - \sigma_0^*) I_{n+1}'(m\alpha)$$

$$\alpha_{31} = m\alpha n(\mu_0 - \sigma_0^*) I_n(m\alpha), \quad \alpha_{32} = m^2\alpha^2 I'_n(m\alpha)$$
$$\alpha_{33} = 2\mu \frac{\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0} I'_n(m\alpha) + (2\mu_0 - \sigma_0^*)[I_{n+1}(m\alpha) + m\alpha I'_{n+1}(m\alpha)]$$

Let us consider the rod buckling mode when $n = m = 1$. For the long-wavelength buckling mode (long rod) $\alpha < 1$. In this case we obtain [4] the characteristic determinant with two-term accuracy in the form

$$\delta = \frac{2\mu_0 - \sigma_0^*}{\lambda_0 + \mu_0} \frac{\alpha^9}{16R^7} \Big\{[\mu_0(\lambda_0 + \sigma_0^*)(\mu_0 + 2\sigma_0^*) + \mu_0(\mu_0 + \sigma_0^*) \times (\lambda_0 + 2\mu_0) - (\lambda_0 + \sigma_0^*)(\lambda_0 + \mu_0)(2\mu_0 - \sigma_0^*)] + \frac{\alpha^2}{24}[2\mu_0(\lambda_0 + \sigma_0^*) \times (12\mu_0 + 11\sigma_0^*) + 2\mu_0(\lambda_0 + 2\mu_0)(9\mu_0 + 7\sigma_0^*) - (2\mu_0 - \sigma_0^*)(\lambda_0 + \mu_0)(12\mu_0 + 9\sigma_0^*)]\Big\} \quad (4.4)$$

Let us examine examples for elastic bodies with potentials of specific form.

E x a m p l e 1. Within the framework of the second variant of the theory of small precritical deformations, let us examine the example for a body with an elastic potential in the form (3.19). In this case we obtain from (3.19), (3.20) and (4.4)

$$p_c \approx {}^1/_2 p_e, \quad p_e = {}^1/_4 \alpha^2 \mu (3\lambda + 2\mu)/(\lambda + \mu) \quad (4.5)$$

Here $p_e$ is the Euler force under axial-compression for a circular rod.

E x a m p l e 2. Within the framework of the theory of finite precritical deformations let us consider an example for a body with a potential of harmonic type (3.22). In this case, the relationships (3.23) hold. From (3.22), (3.23) and (4.4), we obtain for the long-wavelength buckling mode

$$(\lambda_1)_c \approx 1 - {}^1/_8 \alpha^2 \mu / (\lambda + \mu) \quad (4.6)$$

Let $p^*$ denote the compressive load intensity per unit area at the time of buckling. It follows from [2]

$$p^* = -\sigma_0^* \lambda_1^{-1} \quad (4.7)$$

From the third expression in (3.23), (4.6) and (4.7) we obtain

$$p_c^* \approx {}^1/_2 p_e \quad (4.8)$$

The expressions (4.8) and (4.5) agree. Therefore, if the critical load is measured per unit area at the time of buckling, then the results obtained by the theory of finite precritical deformations and by the second variant of the theory of small precritical deformations agree in the case of the long-wavelength buckling mode (for a long rod). Analogous results are obtained for strips and plates.

The results obtained afford the possibility of making a general deduction about the stability problem for uniform all-around compression of simply-connected isotropic compressible bodies on one part $S_2$ of whose surface $S = S_1 + S_2$ hinge-support or rigid clamping conditions are given. This deduction is that the equilibrium state will be stable if the pressure is applied in the form of a follower load on the part $S_1$ of the body surface, and unstable if the pressure is applied in the form of a dead load on the part $S_1$ of the body surface. In this latter case, the critical load for thin-walled bodies is half the Euler force under compression.

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